Ordered phase of the one-dimensional Ising spin glass with long-range interactions

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The one-dimensional long-range Ising spin glass provides useful insights into the properties of finite-dimensional spin glasses with short-range interactions. The defect energy renormalization-group equations derived for it by Kotliar, Anderson, and Stein have been reexamined and a new fixed point has been found. The fixed points which have previously been studied are found to be inappropriate. It is shown that the renormalization-group equations themselves directly imply that that the spin-glass phase is replica symmetric, in agreement with droplet model expectations, when the range exponent σ lies between the upper and lower critical values.

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I. INTRODUCTION

In recent years there have been many simulations of one-dimensional Ising spin glasses with long-range interactions.[1–6] The Hamiltonian of these studies is usually a variant of the following:

$$H = -\sum_{\langle ij \rangle} J_{ij} S_i S_j, \tag{1}$$

where the Ising spin S_i takes on the values ± 1, the sum is over all pairs $\langle ij \rangle$, and i and j are positions on the one-dimensional lattice. The interaction

$$J_{ij} = J \frac{\epsilon_{ij}}{|i-j|^{\sigma}}, \tag{2}$$

where the ϵ_{ij} are independent random variables with a Gaussian distribution of zero mean and unit variance. This model was introduced by Kotliar, Anderson, and Stein,[7] to be referred to as KAS, who showed that for $\sigma < 2/3$ the model has mean-field critical exponents and nonmean-field exponents for $2/3 < \sigma < 1$. When $\sigma > 1$, there is no finite-temperature phase transition. Thus the σ interval, $2/3 < \sigma < 1$, is the analog for short-range spin glasses of the dimension range between the upper critical dimension ($d_u = 6$) and the lower critical dimension. A rough correspondence between a value of σ in the long-range one-dimensional model and the space dimension d in the short-range model has been suggested[6,8]

$$d = \frac{2 - \eta(d)}{2\sigma - 1}, \tag{3}$$

where $\eta(d)$ is the critical exponent of the short-range model.

One motivation for studies of the one-dimensional model with long-range interactions is that by just changing the value of σ one can explore both systems corresponding to high and low dimensionality. Thus study of this long-range one-dimensional model can cast light on the long-standing expectation that the whole nature of the short-range spin-glass state changes at six dimensions.[9] It is impossible to imagine doing a good simulation for a ten-dimensional system but simulations at the corresponding value of σ, which is 0.6, are no harder than those at, say, $\sigma = 0.75$, which corresponds to a dimension less than 6.

Above six dimensions the ordered phase is expected to be described by Parisi's replica symmetry-breaking procedure.[10] There the standard procedure of performing the loop expansion around the mean-field solution should suffice. Below six dimensions it has been argued that the spin-glass phase is replica symmetric and described by the droplet picture.[11–13] Thus above six dimensions, there is expected to be a de Almeida-Thouless (AT) transition in a magnetic field but none is expected below six dimensions. In the one-dimensional model, strong simulational evidence was found for an AT transition for $\sigma = 0.6$ but at $\sigma = 0.75$ there was no sign of an AT transition.[6] Unfortunately, as so often with numerical studies, it is possible to find another way of analyzing the data to reach the opposite conclusion.[14] What is needed therefore is an analytical approach which can settle the long-running debate between the simulators.

For these long-range one-dimensional models KAS found an analytic treatment which should be quantitative near the lower critical value of σ, which is 1. [KAS also studied critical behavior in the vicinity of the "upper critical value of σ," (which is 2/3) by means of conventional perturbative expansions in $(\sigma - 2/3)$]. In this paper we shall follow their second renormalization-group (RG) procedure, which involves an extension of equations given by Cardy,[15] to obtain an expansion about the "lower critical value." Expansions about the lower critical dimension have never been done for short-range spin glasses, probably because not even the precise value of the lower critical dimension is known for them. For the long-range one-dimensional model, KAS derived the RG flow equations which describe the flows of kink (or defect or droplet) couplings and their associated fugacities not only near the transition temperature T_c but also throughout the low-temperature phase. Thus the solution of these equations should allow us to determine not only the values of the critical exponents but also the nature of the ordered phase. We shall employ a technique used previously[16] to show that the spin-glass phase has replica symmetry when $2/3 < \sigma < 1$. This is a direct consequence of the KAS equations themselves and does not require us to find an explicit solution of them, which is fortunate as we are unable to provide one.

The RG equations of KAS involve the replica procedure and are very difficult to solve. In Sec. II it is shown that the fixed-point solution found by KAS themselves is not the appropriate fixed point—it actually is relevant to a multicritical point—and does not describe the behavior across the paramagnet-spin-glass phase boundary. Khurana[17] introduced important ideas which simplify the task of solving the KAS RG equations. The calculations of this paper in Secs. II and III are an extension of them. The actual fixed point found by Khurana is not the correct one when the number of replicas $n<2$; in the replica trick we have to take the limit $n\to 0$. In Sec. III a solution of the KAS RG equations is obtained which is valid for the limit $\sigma\to 1$ and when $T\le T_c$.

In Sec. IV the RG flow equations are solved when a small random magnetic field is included in the Hamiltonian and the magnetic critical exponent is obtained. In Sec. V it is shown that the spin-glass phase has replica symmetry for $2/3<\sigma<1$. This is the main result of the paper.

II. RENORMALIZATION-GROUP EQUATIONS

We begin with a brief outline of the procedure of KAS by which they obtained an expansion around the lower critical value of $\sigma(=1)$. They used the replica method to write the average over the ϵ_{ij} of the nth power of the partition function as

$$\overline{Z^n} = \mathrm{Tr}\,\exp[-\beta H_n]$$

$$=\mathrm{Tr}_{S_i^\alpha}\exp\left(\frac{1}{2}\beta^2J^2\sum_{i<j}\sum_{\alpha,\beta}S_i^\alpha S_i^\beta S_j^\alpha S_j^\beta/|i-j|^{2\sigma}\right). \tag{4}$$

The free energy

$$F=-(\beta n)^{-1}(\overline{Z^n}-1), \tag{5}$$

as $n\to 0$. At each site i there are the replicated spin variables S_i^α, where $\alpha=1,2,\ldots,n$. The possible values of S_i^α lie at the vertices of an n-dimensional hypercube. KAS introduced n-component vectors $\boldsymbol{\sigma}_a$, $a=1,\ldots,2^n$ to describe the 2^n different values of $\{S_i^\alpha\}$, i.e., $\boldsymbol{\sigma}_1=(1,1,\ldots,1,1)$, $\boldsymbol{\sigma}_2=(1,1,\ldots,1,-1)$, $\boldsymbol{\sigma}_3=(1,1,\ldots,-1,-1)$, etc. In the following, replica indices which run from $1,2,\ldots,n$ will be denoted by Greek symbols such as α or β. Labels which run over the 2^n spin states will be denoted by Roman labels such as a or b. The interaction energy between a spin in state a at site i and a spin in state b at site j is

$$\frac{K_{ab}}{|i-j|^{2\sigma}}=\frac{1}{2}\beta^2J^2\frac{[(\boldsymbol{\sigma}_a\cdot\boldsymbol{\sigma}_b)^2-n^2]}{|i-j|^{2\sigma}}, \tag{6}$$

where a constant, n^2, has been subtracted to ensure that $K_{aa}\equiv 0$. KAS wrote the replicated Hamiltonian in terms of kink or defect variables; a kink of type ab is at site i if that site is in state $\boldsymbol{\sigma}_a$ and site $i+1$ is in state $\boldsymbol{\sigma}_b$. For each kink type ab there is an associated fugacity Y_{ab}, and $Y_{aa}\equiv 0$. KAS showed that a change in the lattice spacing $a\to ae^l$ can be compensated by a change in the kink fugacities Y_{ab} and kink couplings K_{ab} to leave invariant the replicated averaged partition function of Eq. (4). This leads to the RG flow equations

$$\frac{dY_{ab}}{dl}=Y_{ab}\left(1+\frac{2K_{ab}}{2\sigma-1}\right)+\sum_{c\neq a,b}Y_{ac}Y_{cb}, \tag{7}$$

$$\frac{dK_{ab}}{dl}=2(1-\sigma)K_{ab}-\sum_c Y_{ac}^2(K_{ab}-K_{bc}+K_{ac})$$
$$-\sum_c Y_{bc}^2(K_{ab}-K_{ac}+K_{bc}). \tag{8}$$

These equations are exact for small fugacities. When σ is close to 1, the effect of the fugacities does indeed turn out to be small. As σ decreases toward the "upper critical value" 2/3, higher terms in Y_{ab} will be needed. But the expectation is that the model for σ just below 1 is qualitatively similar to a model with σ just above 2/3. Thus if one can establish that the spin-glass phase is replica symmetric using Eqs. (7) and (8), (as we can), then this behavior should hold up to the upper critical value 2/3.

One normally proceeds by starting off the RG flows from the bare values of the kink couplings K_{ab} and the corresponding bare fugacities Y_{ab}. The initial value of the kink couplings K_{ab}

$$K_{ab}(l=0)=\frac{\beta^2J^2}{2}k(p), \tag{9}$$

where using Eq. (6)

$$k(p)=[(n-2p)^2-n^2]=-4p(n-p). \tag{10}$$

Here p is the number of replicas in which spins at two adjacent sites, in state a at i and b at $i+1$, are antiparallel and takes the values $p=1,2,\ldots,n$.

One can use symmetry considerations to simplify the $2^n\times 2^n$ matrices K_{ab} and Y_{ab}. Renumbering the replicas should make no physical difference. Formally permutations π of the permutation group $\mathcal{S}_n$ acting on the replica indices $1,\ldots,n$ must result in

$$K[\pi(\boldsymbol{\sigma}),\pi(\boldsymbol{\sigma}')]=K(\boldsymbol{\sigma},\boldsymbol{\sigma}')\quad\text{for every}\quad \pi\in\mathcal{S}_n. \tag{11}$$

As the bare couplings in Eq. (6) are of the form $K(\boldsymbol{\sigma},\boldsymbol{\sigma}')=K(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}')$ we shall suppose that this feature remains true of the renormalized kink couplings; it is preserved by the RG flow equations of Eqs. (7) and (8). An example might make clearer the simplification which these symmetries allow. Consider the case of $n=2$. Then the matrix K_{ab} is a 4×4 matrix. A state like a corresponds to one of 00, 01, 10, and 11 where a down spin is represented by 0 rather than -1 to ease layout. The elements of K_{ab} are

$$K_{00,00}=0,\quad K_{00,01}=s,\quad K_{00,10}=s,\quad K_{00,11}=d,$$

$$K_{01,00}=s,\quad K_{01,01}=0,\quad K_{01,10}=d,\quad K_{01,11}=s,$$

$$K_{10,00}=s,\quad K_{10,01}=d,\quad K_{10,10}=0,\quad K_{10,11}=s,$$

$$K_{11,00}=d,\quad K_{11,01}=s,\quad K_{11,10}=s,\quad K_{11,11}=0,$$

where s denotes the value of K_{ab} when a and b differ in a single replica and d denotes its value if a and b differ in two replicas. Thus a 16 element matrix only depends on two parameters because of the symmetries. In general, the $2^n\times 2^n$ matrix K_{ab} has just n independent entries after taking the permutation symmetries into account. These can be parametrized by $K(p)$, where $K(p)$ is the value of K_{ab} when the states a and b differ in p replicas and $p=1,2,\ldots,n$. There is a similar simplification for Y_{ab} which can be parametrized by $Y(p)$ if the states a and b differ in p replicas. In the given example of the 4×4 matrix, (the case of $n=2$), $s=K(1)$ and $d=K(2)$.

The $2^n\times 2^n$ matrix K_{ab} is readily diagonalized.[18,19] In general there are $n+1$ distinct eigenvalues, whose degeneracies are $1,n,n(n-1)/2,\ldots,\binom{n}{p},\ldots,1$. The sum of these degeneracies is 2^n as required. There is a trivial eigenvalue equal to $\sum_{p=1}^{n}\binom{n}{p}K(p)$ of degeneracy unity. If one sets $K(p)\propto[(n-2p)-n]$, then there is one other nonzero eigenvalue of degeneracy n and its eigenvectors identify it as corresponding to a replicated ferromagnetic Hamiltonian. If $K(p)\propto k(p)$, as defined in Eq. (10), there is again only one nontrivial nonvanishing eigenvalue, of degeneracy $n(n-1)/2$, which would be natural for the spin-glass state. Note that for ferromagnets the order parameter in the replicated Hamiltonian is $\langle S_i^\alpha\rangle$, which has n components while the spin-glass order parameter, $\langle S_i^\alpha S_i^\beta\rangle$, has $n(n-1)/2$ independent components.

The fixed point studied by KAS of Eqs. (7) and (8) was $K_{ab}=K^*$ and $Y_{ab}=Y^*$ for all $a\neq b$. The degeneracy of the nontrivial eigenvalue of K_{ab}^* is 2^n-1. This fixed point therefore corresponds to a multicritical fixed point at which all the order parameters such as $\langle S_i^\alpha\rangle$, $\langle S_i^\alpha S_i^\beta\rangle$, $\langle S_i^\alpha S_i^\beta S_i^\gamma\rangle$, etc., go critical simultaneously. The critical exponents associated with this fixed point are those of the corresponding long-range Q-state Potts model,[17] with $Q=2^n$, (where as usual n is set to zero at the end of the calculation). It is clearly not the appropriate fixed point for the study of the paramagnetic to spin-glass transition.

There is a further additional symmetry in the problem, that of time-reversal symmetry.[17] Notice that the bare kink coupling of Eq. (10) is left unchanged by the exchange $p\to n-p$. This is a manifestation of the invariance of K_{ab} in Eq. (6) under the time-reversal transformation $\boldsymbol{\sigma}\to-\boldsymbol{\sigma}$. This symmetry of the initial couplings is again maintained by the RG flow equations. Since $K(0)=0$ by construction, time-reversal invariance implies that $K(n)=0$ and in general that $K(p)=K(n-p)$. Time-reversal symmetry implies that the fugacities obey the relation $Y(p)=Y(n-p)$; in particular, that $Y(n)=0$ as $Y(0)=0$. We could, in principle, reduce further the number of independent kink coupling and their associated fugacities using this symmetry but it is more convenient not to do this.

Khurana[17] reexpressed the RG equations of Eqs. (7) and (8) as RG equations for $K(p,l)$ and the associated $Y(p,l)$. He found that by setting $K(p,l)]=K(l)k(p)$, the RG equations for $K(p,l)$ reduce to just a single equation for $K(l)$

$$\frac{dK(l)}{dl}=\left[2(1-\sigma)-8\sum_{r=1}^{n-1}\binom{n-2}{r-1}Y(r,l)^2\right]K(l), \quad (12)$$

with the initial value $K(l=0)=\beta^2J^2/2$. The RG flow equations for $Y(p,l)$ are more complicated

$$\frac{dY(p,l)}{dl}=Y(p,l)\left[1+\frac{2K(l)k(p)}{2\sigma-1}\right]+\sum_{m=0}^{p}\binom{p}{m}\sum_{r=0}^{n-p}\binom{n-p}{r}Y(m+r,l)Y(r+p-m,l). \quad (13)$$

Note that $Y(0,l)=Y(n,l)\equiv 0$ and $p=1,2,\ldots,n-1$. The initial values are

$$Y(p,l=0)=\exp[U(0)K(l=0)k(p)], \quad (14)$$

where, following Cardy,[15] $U(0)\approx 1+\gamma+\beta^2J^2/2$. $\gamma\approx 0.577$ is Euler's constant.

Khurana found a fixed point of these equations which is probably the correct solution for $n>2$. There is a fixed point where $Y^*(1)=Y^*(n-1)$ is of order $(1-\sigma)^{1/2}$ and the other $Y^*(p)$ are of order $(1-\sigma)$ or higher powers. The critical temperature for $\sigma\to 1$ was given by

$$1+\frac{2K^*k(1)}{2\sigma-1}\approx 1+[(n-2)^2-n^2]\beta_c^2J^2=0. \quad (15)$$

The transition temperature $T_c=2(n-1)^{1/2}J$ becomes pure imaginary when $n<1$. Hence this fixed point of Eqs. (12) and (13) cannot be the appropriate fixed point for the spin glass where we have to take $n=0$.

As they stand the RG equations do not seem to make any sense. They hold only for small fugacities $Y(p,l)$; their derivation neglects terms of $O(Y^4)$.[15] But the initial values of these fugacities, $\exp[U(0)\beta^2J^2k(p)/2]$ is extremely large in the low-temperature limit where β goes to infinity. The key to progress is to focus on the sums over fugacities in equations such as Eqs. (12) and (13). By virtue of the limit $n\to 0$, these individually large terms when summed give rise to small well-behaved expressions. As a consequence we can find solutions of the KAS RG equations in the regime in which they are expected to be accurate, that is, for σ approaching unity. This is the subject of the next section.

III. SOLVING THE RG FLOW EQUATIONS

We start by solving Eq. (13) for $Y(p,l)$. We will make a guess that the solutions are of the form

$$Y(p,l)=y(l)\exp[A(l)^2k(p)/8], \quad (16)$$

when p is $1,2,\ldots,n-1$, $y(0)=1$, and

$$\frac{A(l)^2}{8}=\frac{2}{2\sigma-1}\int_0^l dl'K(l')+\frac{1}{2}\beta^2J^2U(0). \quad (17)$$

Substituting this into Eq. (13) gives after some algebra, which is done in Appendix,

$$\frac{dy(l)}{dl}=y(l)-4y^2(l). \quad (18)$$

Corrections to it are exponentially small, of $O(\exp[-1/(1-\sigma)^2])$ (see Appendix) and so can be dropped in the limit $\sigma\to 1$. It has solution

$$y(l) = \frac{1}{4 - 3e^{-l}}, \tag{19}$$

showing that $y(l)$ decreases from its initial value of unity to its fixed-point value $y^* = 1/4$.

We now turn to Eq. (12) for the RG flow of $K(l)$. We shall substitute the expression $Y(p,l)$ from Eq. (16) into Eq. (12), and set $n=0$ in $k(r)$ at the outset (keeping it in to the end and then setting it to zero gives the same result but lengthens the equations). For $n=0$, $k(r)=4r^2$. We now need to evaluate the sum

$$S = \sum_{r=1}^{n-1} \binom{n-2}{r-1} \exp[A(l)^2 r^2]. \tag{20}$$

An integral representation allows S to be rewritten as

$$S = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \sum_{r=1}^{n-2} \binom{n-2}{r-1} \exp[\sqrt{2}A(l)xr - x^2/2] \tag{21}$$

which permits the sum over r to be performed. Then

$$S = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \exp[\sqrt{2}A(l)x - x^2/2](1 + \exp[\sqrt{2}A(l)x])^{n-2} \tag{22}$$

which becomes after setting once more n to zero and some rearrangement

$$S = \frac{1}{4}\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \exp[-x^2/2]\mathrm{sech}^2[A(l)x/\sqrt{2}]. \tag{23}$$

We shall proceed upon the assumption that $A(l)$ is very large. Below we will show that its minimum value [see Eq. (35)] is of $O[1/(1-\sigma)]$. For large $A(l)$, in the integral over x in Eq. (23), $\exp[-x^2/2]$ can be replaced by unity because the sech term rapidly becomes very small for values of $x \sim 1/A(l)$. The integral can now be evaluated directly giving

$$S = \frac{1}{2\sqrt{\pi}A(l)}. \tag{24}$$

Our final expression for the RG flow of $K(l)$ is

$$\frac{dK(l)}{dl} = \left[2(1-\sigma) - \frac{4y(l)^2}{\sqrt{\pi}A(l)}\right]K(l), \tag{25}$$

with the initial condition that $K(0) = \beta^2 J^2/2$.

We have thus reduced the RG equations to an integrodifferential equation. [The integral is in the definition of $A(l)$.] Equation (25) could of course be solved numerically. However, there is a simplification which allows for an analytical solution when σ is close to unity. Equation (19) shows that $y(l)$ approaches its fixed point as a function of e^{-l}. However, we shall see that $K(l)$ varies with l much more slowly—as a function not of l but the combination $2(1-\sigma)l$. Since Eq. (25) is itself only valid for σ close to unity, we can proceed by setting in it $y(l)$ to its fixed-point value 1/4. This is because $y(l)$ will effectively be at its fixed-point value before $K(l)$ has started to flow. Then Eq. (25) becomes analytically tractable. We shall also set $2\sigma-1$ which occurs in Eq. (17) to 1 which is correct to leading order in $1-\sigma$.

Let $z(l) = A(l)^2/8$. $z(0) = \beta^2 J^2 U(0)/2$. The first derivative

$$\frac{dz(l)}{dl} \equiv \dot{z} = 2K(l) \tag{26}$$

and $\dot{z}(0) = \beta^2 J^2$. Its second derivative

$$\ddot{z}(l) = 2\frac{dK(l)}{dl} = 2(1-\sigma)\dot{z}(l) - \frac{\dot{z}(l)}{\sqrt{128\pi z(l)}}. \tag{27}$$

This equation can be integrated to

$$\dot{z}(l) = 2(1-\sigma)z(l) - \sqrt{z(l)/32\pi} + C \tag{28}$$

with the constant of integration

$$C = \beta^2 J^2 - (1-\sigma)\beta^2 J^2 U(0) + \sqrt{\beta^2 J^2 U(0)/64\pi}. \tag{29}$$

Equation (28) can be regarded as a quadratic equation for $\sqrt{z(l)}$, with solution

$$\sqrt{z(l)} = \frac{1 \pm \sqrt{1 - Q[C - 2K(l)]}}{\sqrt{512\pi}(1-\sigma)} = \frac{1 \pm \sqrt{(1 - T_c/T)^2 + 2Q[K(l) - K(0)]}}{\sqrt{512\pi}(1-\sigma)}, \tag{30}$$

where

$$T_c = 16\sqrt{\pi U(0)}(1-\sigma)J, \tag{31}$$

and $Q = 256\pi(1-\sigma)$. As a consequence of these manipulations we now have an ordinary differential equation for $K(l)$,

$$\frac{dK(l)}{dl} = 2(1-\sigma)K(l) \times \left[1 - \frac{1}{1 \pm \sqrt{(1 - T_c/T)^2 + 2Q[K(l) - K(0)]}}\right]. \tag{32}$$

Inspection of this equation shows that $K(l)$ will indeed be a function of $2(1-\sigma)l$, which justifies our setting in Eq. (25) $y(l)$ to its fixed-point value 1/4.

It may be more intuitive to express the flow equation for $K(l)$ as a flow equation for $T(l)$, the effective temperature on the length scale ae^l, defined by setting

$$K(l) = \frac{J^2}{2T(l)^2}. \tag{33}$$

Then

$$\frac{dT(l)}{dl} = -(1-\sigma)T(l) \times \left[1 - \frac{1}{1 \pm \sqrt{(1 - T_c/T)^2 + QJ^2[1/T(l)^2 - 1/T^2]}}\right] \tag{34}$$

with the initial condition that $T(0) = T$.

Fixed points may arise if $dT(l)/dl=0$. For $T<T_c$ we take the solution with the plus sign while for $T>T_c$, we take the solution with the minus sign. There is a fixed point at $T(l)=0$, the zero-temperature fixed point. The *unstable* critical fixed point is at $T=T_c$ and $T(l)=T_c$. If $T<T_c$, $T(l)$ flows from T to the zero-temperature sink $T(l)=0$. When $T>T_c$, $T(l)$ increases with l but eventually it reaches a value at which the argument of the square-root sign in Eq. (34) becomes zero and beyond that point, $T(l)$ goes complex. We conclude that our solution has no validity for $T>T_c$. From now on we will always take $T\leq T_c$.

The derivation of the RG equation for $K(l)$ required $A(l)$ to be large. From Eq. (30) the minimum value of $A(l)$ occurs at $l=0$ when $T=T_c$,

$$A(0)\to\frac{1}{8\sqrt{\pi}(1-\sigma)}. \tag{35}$$

This is indeed large in the limit when $\sigma\to 1$ but if σ is fixed at some value close to unity, say 0.95, $A(l)$ will *not* be large enough to make the approximation procedure reliable near and above T_c. However, we can work at a fixed value of σ when $T\ll T_c$. Then $A(0)$ becomes very large

$$A(0)\approx\frac{T_c}{8\sqrt{\pi}(1-\sigma)T} \tag{36}$$

and our solution will become essentially exact. Furthermore, as $A(l)$ increases to infinity as $l\to\infty$ when $T<T_c$, then our calculations of quantities such as $\mathcal{S}(l)$, defined below, which are directly related to $A(l)$, become essentially exact when l is large.

Fortunately some results can be obtained without needing an explicit solution of the RG equations. Equation (12) can be rewritten as a flow equation for $T(l)$,

$$\frac{dT(l)}{dl}=-T(l)[1-\sigma-\mathcal{S}(l)], \tag{37}$$

where the sum $\mathcal{S}(l)$ is defined to be

$$\mathcal{S}(l)=4\sum_{r=1}^{n-1}\binom{n-2}{r-1}Y(r,l)^2. \tag{38}$$

Our calculation of the sum $\mathcal{S}(l)$ gave $\mathcal{S}(l)=2y(l)^2/[\sqrt{\pi}A(l)]$, which approaches zero as $l\to\infty$ when $T<T_c$. This large l result should be exact even for fixed values of σ. It is fundamentally a consequence of the fact that at low temperatures there will be few large thermally excited droplets. Hence in the vicinity of the zero-temperature fixed point,

$$\frac{dT(l)}{dl}\to-(1-\sigma)T(l)=-\theta T(l)\quad\text{as}\quad l\to\infty \tag{39}$$

so the droplet scaling exponent $\theta=1-\sigma$, in accordance with conventional expectations.[2,3,13,20] In fact one expects this result to hold for values of σ not just close to 1 but throughout the range $2/3<\sigma<2$.

The transition temperature T_c can be determined from Eq. (31) by solving the quadratic equation which arises from the temperature dependence of $U(0)$. The result is

$$\left[\frac{T_c}{J}\right]^2=(1+\gamma)Q(1-\sigma)/2+\sqrt{(1+\gamma)^2Q^2(1-\sigma)^2+2Q(1-\sigma)}/2. \tag{40}$$

As $\sigma\to 1$, $T_c\to\sqrt{\sqrt{128\pi}(1-\sigma)}J$. Thus no finite temperature transition is expected right at $\sigma=1$. This is to be contrasted with what happens for long-range Ising ferromagnets and Potts models which do have a finite-temperature transition at their own lower critical value of σ.[15] The difference occurs because at $T=0$ in an Ising ferromagnet all spins are aligned. No kinks are present in the low-temperature phase. They appear for $T>T_c$ when their entropy overcomes the energy cost of their creation. In the leading term in $\overline{Z^n}$ as $T\to 0$, the replicas at all sites have their spins aligned.[17] The energy of this state varies as n^2 so does not contribute to the actual physical free energy. The leading term for the free energy contains flipped spins and kinks,[17] and gives a contribution to the free energy of $O(n)$. The kink fugacities are always nonzero at all temperatures in the spin glass.

The RG equations for the case of $\sigma=1$ are as in Eqs. (25), (28), and (29), with σ there set to unity. The resulting flow equation for $K(l)$ is

$$\frac{dK(l)}{dl}=-\frac{K(l)}{64\pi[\beta J\sqrt{U(0)/64\pi}+2[K(0)-K(l)]]}. \tag{41}$$

The flow starts from $K(0)=\beta^2J^2/2$ which is large at low temperatures. We can determine the correlation length $\xi(T)$ by finding the scale l^* at which $K(l)$ is of order unity. Then $\xi(T)=ae^{l^*}$. By integrating up Eq. (41) one obtains to leading order

$$\frac{\xi(T)}{a}\sim\exp\{64\pi[\beta^2J^2+\beta J\sqrt{U(0)/(64\pi)}]\ln\beta^2J^2\}. \tag{42}$$

Thus as $T\to 0$, $\xi(T)$ diverges extremely rapidly.

IV. BEHAVIOR IN A MAGNETIC FIELD

It is instructive to study the behavior of the system in a random magnetic field and, in particular, the magnetic exponent y_h using the RG formalism. This exponent describes the scaling dimension of the field conjugate to the spin-glass order parameter. Such a field is the Gaussian random field h_i of zero mean and standard deviation h. Its presence changes the Hamiltonian to

$$H=-\sum_{\langle ij\rangle}J_{ij}S_iS_j-\sum_i h_iS_i. \tag{43}$$

After replicating and averaging over the couplings J_{ij} and the fields h_i, an additional term appears in the replicated Hamiltonian βH_n of Eq. (4)

$$-\frac{1}{2}\beta^2h^2\sum_i\left(\sum_{\alpha=1}^{n}S_i^\alpha\right)^2. \tag{44}$$

Let $\boldsymbol{t}$ denote the n-component vector $(1,1,1,\ldots,1)$. $\boldsymbol{t}$ can be used to rewrite the magnetic field term in the replicated Hamiltonian as

$$-\frac{1}{2}N\beta^2h^2n-\sum_i\frac{1}{2}\beta^2h^2\{[\boldsymbol{t}\cdot\boldsymbol{\sigma}_a(i)]^2-n\}. \tag{45}$$

It is convenient to express the nontrivial term in Eq. (45) in terms of $H_a(i)\equiv\mathcal{H}\{[\boldsymbol{t}\cdot\sigma_a(i)]^2-n\}$, where $\mathcal{H}=\beta^2h^2/2$. $\mathcal{H}$ is the conjugate field for

$$\frac{1}{N}\sum_i\sum_{\alpha\neq\beta}\langle S_i^\alpha S_i^\beta\rangle, \tag{46}$$

the spin-glass order parameter. The magnetic exponent $y_{\mathcal{H}}$ is related to the critical exponent η via

$$y_{\mathcal{H}}=(d+2-\eta)/2 \tag{47}$$

in dimension d. With the long-range interactions we are studying, $2-\eta=2\sigma-1$,[21] so $y_{\mathcal{H}}=\sigma$. The question of interest is whether this result, which is valid for $2/3<\sigma<1$, can be recovered from the KAS RG equations.

In a previous paper[16] I derived the RG equations satisfied by the H_a when they are small enough so that higher terms in H_a can be neglected;

$$\frac{dH_a(l)}{dl}=H_a(l)+\sum_b Y_{ab}(l)^2[H_b(l)-H_a(l)], \tag{48}$$

for $a=1,\ldots,2^n$. Remarkably these 2^n equations reduce to a single equation for $\mathcal{H}(l)$ when $Y_{ab}(l)$ is expressed in terms of $Y(r,l)$

$$\frac{d\mathcal{H}(l)}{dl}=\left[1-4\sum_{r=1}^{n-1}\binom{n-2}{r-1}Y(r,l)^2\right]\mathcal{H}(l)$$
$$=[1-\mathcal{S}(l)]\mathcal{H}(l). \tag{49}$$

This reduction to a single equation does not require any assumptions on the r dependence of the $Y(r,l)$; it is a general result, of similar origin to the reduction in the RG equations for $K_{ab}(l)$ to the single equation for $K(l)$ in Eq. (12).

We first encountered the sum $\mathcal{S}(l)$ in Eq. (37). Its value at the critical fixed point is $1-\sigma$ when $l=0$. This will be true generically and not just for the limit $\sigma\to1$. Then at $T=T_c$, the RG equation for $\mathcal{H}(l)$ reduces to

$$\frac{d\mathcal{H}(l)}{dl}=\sigma\mathcal{H}(l)\quad\text{as}\quad l\to0, \tag{50}$$

implying that $y_{\mathcal{H}}=\sigma$, as anticipated. One would expect this result to hold for $2/3<\sigma<1$. It is reassuring that we can recover the exact value for the magnetic exponent $y_{\mathcal{H}}$ over this interval as our argument in Sec. V for replica symmetry in the same range proceeds along similar lines.

As we have noted before, $\mathcal{S}(l)$ approaches zero when $T<T_c$ as $l\to\infty$, when the RG flow is to the zero-temperature fixed point. Hence

$$\frac{d\mathcal{H}(l)}{dl}=\mathcal{H}(l)\quad\text{as}\quad l\to\infty, \tag{51}$$

which means that on length scale $L=ae^l$ because $\mathcal{H}=\beta^2h^2/2$, the effective field on a droplet of size L, $h(L)$, scales as $L^{d/2}$, with $d=1$, as expected in Refs. 11–13. This result will hold for all $\sigma>2/3$.

V. PARISI OVERLAP FUNCTION $P(q)$

Whether the ordered phase of the spin glass is replica symmetric or not is determined by the form of the Parisi overlap function $P(q)$. Parisi[22] showed that this could be calculated via

$$P_J(q)=\left\langle\delta\left(q-\frac{1}{N}\sum_i T_iS_i\right)\right\rangle, \tag{52}$$

where the thermal average is over the doubly replicated Hamiltonian $H\{T_i\}+H\{S_i\}$. It is convenient to study

$$F_J(y)=\left\langle\left(\exp\frac{y}{N}\sum_i T_iS_i\right)\right\rangle \tag{53}$$

as

$$P_J(q)=\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}e^{-yq}F_J(y). \tag{54}$$

The bond average, $\overline{F_J(y)}[=F(y)]$, leads to the determination of the Parisi overlap function; $P(q)=\overline{P_J(q)}$. $F(y)$ can be calculated by replicating the spins at each site, $S_i^\alpha,T_i^\alpha,\alpha=1,\ldots,n$. The details of the calculation of $F(y)$ in the context of the kink RG procedure can be found in Ref. 16. The variable $\boldsymbol{\sigma}_a$ is now a vector of $2n$ components, where the first n components contain the $\boldsymbol{S}$ spins while the second n components are the $\boldsymbol{T}$ spins. a now runs from 1 to 2^{2n}. The term in y produces in the replicated Hamiltonian βH_{2n} a term

$$-\sum_i H_a(i)=-\frac{y}{N}\sum_i S_a^1(i)T_a^1(i). \tag{55}$$

S_a^1 and T_a^1 are the first and $(n+1)$th components of $\boldsymbol{\sigma_a}$. The RG flow equations for the $H_a=(y/N)S_a^1T_a^1$ turn out to be identical in form to those in Eq. (48),[16]

$$\frac{dH_a(l)}{dl}=H_a(l)+\sum_b Y_{ab}(l)^2[H_b(l)-H_a(l)] \tag{56}$$

but now $a=1,\ldots,2^{2n}$. There is another remarkable reduction in these 2^{2n} equations to a single equation when Y_{ab} is expressed in terms of $Y(r,l)$. The equations become the single equation for $y(l)$,

$$\frac{dy(l)}{dl}=\left[1-4\sum_{r=1}^{2n-1}\binom{2n-2}{r-1}Y(r,l)^2\right]y(l). \tag{57}$$

The initial condition on this equation is that $y(0)=y/N$. Equation (57) is general; it holds for any $Y(r,l)$.

The sum in Eq. (57) is just the same as that which defines $\mathcal{S}(l)$ in Eq. (38), provided we replace n in that equation by $2n$. In the limit of $n\to0$ they will therefore be identical. Integrating

$$y(l)=(y/N)\exp\left[l-\int_0^l dl'\,\mathcal{S}(l')\right]. \tag{58}$$

At the value of $l=l^*$, where $e^{l^*}=N$, there are just two spins S and T left in the system; the other spins have been integrated out. Their interaction is $y(l^*)$. Then

$$F(y) = \frac{\mathrm{Tr}_{T,S}\exp[y(l^*)TS]}{\mathrm{Tr}_T(1)\mathrm{Tr}_S(1)} = (e^{yq_{EA}} + e^{-yq_{EA}})/2. \quad (59)$$

Here

$$q_{EA} = \exp\left[-\int_0^{l^*} dl\mathcal{S}(l)\right], \quad \text{as} \quad l^* \to \infty. \quad (60)$$

Using Eq. (54), one obtains

$$P(q) = [\delta(q - q_{EA}) + \delta(q + q_{EA})]/2. \quad (61)$$

Equation (61) means that the spin-glass phase has replica symmetry. It is the main result of this paper. The replica symmetric form arises from the reduction in Eq. (56) to a single equation. It does not require us to take the limit $\sigma \to 1$. We expect it to be valid for $2/3 < \sigma < 1$.

In the limit $\sigma \to 1$ we were able to show that $S(l) = 2y(l)^2/[\sqrt{\pi}A(l)]$. [Do not confuse $y(l)$ with $y(l)$]. On substituting for $y(l)$ its fixed-point value of 1/4 and using Eqs. (19) and (32) one obtains

$$q_{EA} = \exp - \left[\frac{\pi/2 - \tan^{-1}(1/b)}{(T_c/T - 1)b}\right], \quad (62)$$

where $b^2 = QJ^2/(T_c - T)^2 - 1$. When $T \to 0$, $q_{EA} \to 1$, as expected. Less expected is the value of q_{EA} as $T \to T_c^-$. From Eq. (62),

$$q_{EA}(T_c) = \exp[-(2\pi)^{3/4}/16]. \quad (63)$$

The conventional expectation is that as $T \to T_c$, $q_{EA} \sim (1 - T/T_c)^\beta$ so that $q_{EA}(T_c) = 0$. This is what will occur at a fixed value of σ. The finite value of $q_{EA}(T_c)$ might just be a peculiarity of our $\sigma \to 1$ procedure.

VI. DISCUSSION

The chief conclusion of the paper is that the spin-glass state of this long-range one-dimensional Ising spin-glass model is replica symmetric when $2/3 < \sigma < 1$. This suggests that short-range d-dimensional spin glasses will be replica symmetric when $d < 6$.

I could only obtain an *explicit* solution of the KAS RG equations for the limit when $\sigma \to 1$ and $T < T_c$. The procedure for obtaining a solution of these equations for a fixed value of σ close to unity, both above and below T_c remains to be found. Fortunately the argument for replica symmetry does not depend on having an explicit solution but on general properties of the KAS equations.

In order to get the power-law behavior which is expected at a fixed value of σ, the flow equation for $T(l)$ when it is close to T_c will have to have the linearized form

$$\frac{dT(l)}{dl} = \frac{1}{\nu}[T(l) - T_c], \quad (64)$$

where ν is the correlation length exponent. The flow will start from its initial value T. This equation can be used in Eq. (60) when

$$q_{EA}(T) = \exp\left[-\int_T^0 dT(l)\frac{dl}{dT(l)}(1 - \sigma)\right], \quad (65)$$

where we have set $\mathcal{S}(l)$ to its fixed-point value $1-\sigma$ and used the fact that when $T < T_c$, $T(l)$ starts at T and flows to zero. On evaluating the integral

$$q_{EA}(T) = (1 - T/T_c)^\beta, \quad (66)$$

where $\beta = \nu(1-\sigma)$. This is just the result which follows from the scaling relation $\beta = \nu(d-2+\eta)/2 = \nu(1-\sigma)$, when $2-\eta = 2\sigma - 1$.[21]

The flow equation for $T(l)$ given in Eq. (34), which holds when $\sigma \to 1$, is not of the linearized form of Eq. (64) when $T(l) - T_c$ is small. However, as $l \to 0$, it does reduce to

$$\frac{dT(l)}{dl} \to (1 - \sigma)(T - T_c) = (1 - \sigma)[T(0) - T_c]. \quad (67)$$

It is therefore tempting to speculate that

$$\nu = \frac{1}{1 - \sigma}, \quad (68)$$

when σ is close to unity.

Alas, in order to check this speculation, the solution of the KAS RG equations at fixed σ is required.

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APPENDIX: THE RG EQUATIONS FOR THE FUGACITIES

In this appendix we indicate in more detail how the set of $n-1$ RG flow equations for the fugacities, Eq. (13), are reduced by means of the guess in Eq. (16) to just a single simple RG equation, that in Eq. (18), and explain why this is essentially exact when σ is close to 1.

We proceed by substituting the guess in Eq. (16) into Eq. (13). According to the guess if $p=0$ or if $p=n$ $Y(0,l) = Y(n,l) = y(l)$ but these terms at $p=0$ or $p=n$ are zero. Our strategy for solving the equations is to define a function $\tilde{Y}(p,l) = Y(p,l)$ when $p = 1, 2, \ldots, n-1$ and $\tilde{Y}(0,l) = y(l) = \tilde{Y}(n,l)$. Then

$$\frac{d\tilde{Y}(p,l)}{dl} = \tilde{Y}(p,l)\left[1 + \frac{2K(l)k(p)}{2\sigma - 1}\right] - 4y(l)\tilde{Y}(p,l) + \sum_{m=0}^{p}\binom{p}{m}\sum_{r=0}^{n-p}\binom{n-p}{r}\tilde{Y}(m+r,l)\tilde{Y}(r+p-m,l).$$

If the term involving the summations can be ignored, this equation has solution for $\tilde{Y}(p,l)$ as in Eq. (16). Our task is thus to show that the summations produce only a very small correction. Define R as

$$\sum_{m=0}^{p}\binom{p}{m}\sum_{r=0}^{n-p}\binom{n-p}{r}\exp\{A^2[k(m+r)+k(r+p-m)]/8\}.$$

Setting once again $k(p)=4p^2$, with the aid of a double integral representation

$$R=\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{dy}{\sqrt{2\pi}}\sum_{m=0}^{p}\binom{p}{m}\sum_{r=0}^{n-p}\binom{n-p}{r}\exp[A(m+r)x + A(r+p-m)y-(x^2+y^2)/2].$$

The sums over r and m can now be done explicitly when R becomes (after setting n to zero again)

$$\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{dy}{\sqrt{2\pi}}\exp[-(x^2+y^2)/2]\left[\frac{e^{Ax}+e^{Ay}}{1+e^{A(x+y)}}\right]^p.$$

The integrals can be simplified by introducing the new variables $u=(x+y)/2$, and $v=x-y$. Then $R=R_uR_v$ where

$$R_u=\int_{-\infty}^{\infty}\frac{du}{\sqrt{2\pi}}e^{-u^2}\,\mathrm{sech}^p(Au)$$

and

$$R_v=\int_{-\infty}^{\infty}\frac{dv}{\sqrt{2\pi}}e^{-v^2/4}\cosh^p(Av/2).$$

When $A(l)$ is large, as it always at temperatures $T\le T_c$, the integrals for R_s and R_d can be done by steepest descents. R_u is of $O[1/A(l)]$ while $R_v\sim\exp[A(l)^2k(p)/16]$. This means that the corrections to Eq. (18) from R are of order $\exp[-A(l)^2k(p)/16]$. The smallest value of $A(l)$ is of order $1/(1-\sigma)$ according to Eq. (35), so provided that σ is close to 1, these corrections should be negligible.

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